Wavelet Transform

In integral transform,

$$(Tf)(y) = \int_{\Omega} k(x,y)\, f(x)\, dx$$

$\hookrightarrow$ Kernel, $\Omega \subseteq \mathbb{R}$

Here $f(x) \longmapsto F(y)$

if, $k(x,y) = e^{-ixy} \rightarrow$ Fourier transform.

$\hookrightarrow$ One parameter family of function (in y)

Now we take a two parameter family of function:

$$\psi_{a,b}(x) = |a|^{-1/2}\, \psi\left(\frac{x-b}{a}\right)$$

then continuous wavelet transform

$$(W_\psi f)(a,b) = \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}(t)}\, dt$$

$\hookrightarrow$ Complex conjugate of ψ.

Wavelet: wavelet is a function defined on $\mathbb{R}$

and $\int_{\mathbb{R}} |\psi|^2 dx < \infty$ — Square integrable function.

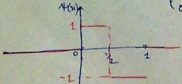
$\hookrightarrow$ This space is denoted by $L^2(\mathbb{R})$

and $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ — admissibility condition.

where $\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, \psi(x)\, dx$.

Note: $\psi \in L^2(\mathbb{R}) \Rightarrow \psi_{a,b} \in L^2(\mathbb{R})$

e.g. Haar wavelet:- $\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases}$



$\int_{\mathbb{R}} |\psi|^2 dx < \infty$

$\Rightarrow$ Square integrable function

i.e. $\psi \in L^2(\mathbb{R})$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left\{ \int_0^{1/2} e^{-i\omega x} dx - \int_{1/2}^{1} e^{-i\omega x} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{e^{-i\omega x}}{i\omega}\Big|_0^{1/2} + \frac{e^{-i\omega x}}{i\omega}\Big|_{1/2}^{1} \right\} = \frac{1}{\sqrt{2\pi}} \left\{ -\frac{e^{-\frac{i\omega}{2}}}{i\omega} + \frac{1}{i\omega} + \frac{e^{-i\omega}}{i\omega} - \frac{e^{-i\omega/2}}{i\omega} \right\}$$

$\Rightarrow \hat{\psi}(\omega) = \frac{4i}{\sqrt{2\pi}} \frac{e^{i\omega/2}}{\omega} \sin^2\frac{\omega}{4} \Rightarrow |\hat{\psi}(\omega)| = \frac{4}{\sqrt{2\pi}} \frac{\sin^2 \omega/4}{\omega}$

$\Rightarrow |\hat{\psi}(\omega)|^2 = \frac{8}{\pi} \frac{\sin^4 \omega/4}{\omega^2}$

and $\int_{-\infty}^{\infty} \frac{|\psi^2(\omega)|^2}{|\omega|} d\omega = \int_{-\infty}^{\infty} \frac{8}{\pi} \frac{\sin^4(\omega/4)}{|\omega|^3} d\omega \leq \int_{-\infty}^{\infty} \frac{8}{\pi} \frac{d\omega}{|\omega|^3} < \infty$

Hence the admissibility condition is satisfied.

→ Here $|\hat{\psi}(\omega)|$ → is even
→ ≥ 0
→ differentiable everywhere except at 0.

Note:
ψ is a wavelet
ϕ is integrable and bounded
$\Rightarrow \psi * \phi$ is also a wavelet.

Q.1: Take $\psi \to$ Haar wavelet, $\phi(x) = e^{-x^2}$ $\Rightarrow \psi * \phi = ?$

– $(W_\psi f)(a,b) = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)dt \to$ ⊛

Continuous wavelet transform is a family of integral transforms. f, ψ should be such that the RHS integral of ⊛ exists.

Note: 1) If $\psi \in L^2(\mathbb{R}) \Rightarrow \psi(a,b) \in L^2(\mathbb{R})$

Proof: $\psi \in L^2(\mathbb{R})$

$\Rightarrow \int_{-\infty}^{\infty} |\psi|^2 dx < \infty$

$\psi(a,b) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right)$

$\Rightarrow \int_{-\infty}^{\infty} |a|^{-1} \left|\psi\left(\frac{x-b}{a}\right)\right|^2 dx$ → Put $\frac{x-b}{a} = t \Rightarrow dx = a\,dt$

$= \frac{1}{|a|} \int_{-\infty}^{\infty} |\psi(t)|^2 . a\,dt < \infty$

2) If ψ is admissible then so also $\psi_{a,b}$.

Proof: To prove: $\int_{-\infty}^{\infty} \frac{|\hat{\psi}_{a,b}(\omega)|^2}{|\omega|} d\omega < \infty$

Now $\hat{\psi}_{a,b}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) . e^{-i\omega x} dx$

$= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} \psi\left(\frac{x-b}{a}\right) e^{-i\omega x} dx$. Put $\frac{x-b}{a} = t$ $\Rightarrow dx = a\,dt$

$$= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-iw(at+b)} \psi(t)\,dt = e^{-iwb} \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-iwat} \psi(t)\,dt$$

$$= e^{-iwb} \sqrt{\frac{a}{2\pi}}\, \hat{\psi}(aw)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{|\hat{\psi}_{a,b}(w)|^2}{|w|}\,dw = \int_{-\infty}^{\infty} \left(\frac{a}{2\pi}\right) \frac{|\hat{\psi}(aw)|^2}{|w|}\,dw$$

Put $aw = w'$

$\Rightarrow a\,dw = dw'$

$$= \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w')|^2\,dw'}{|w'|\,2\pi} \cdot a < \infty$$

$\Rightarrow \psi_{a,b}$ is admissible.

Continuous Wavelet Transform :-

$\psi \in L^2(\mathbb{R})$

$a, b \in \mathbb{R},\ a \neq 0$

$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right)$

$$\Rightarrow \boxed{(W_\psi f)(a,b) = \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}(t)}\,dt} = \langle f, \psi_{a,b} \rangle$$

$\psi \rightarrow$ mother wavelet / analysing wavelet

$a \rightarrow$ scaling factor

$b \rightarrow$ translating factor

Take ψ to be Haar wavelet.

$$\Rightarrow \psi_{1,2} = \begin{cases} 1 & , 2 \leq x \leq 2.5 \\ -1 & , 2.5 \leq x \leq 3 \\ 0 & , \text{otherwise} \end{cases}$$

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Q1: If ψ is a wavelet and ϕ is bounded integrable function then show that $\psi * \phi$ is a wavelet.

Ans.1: **Step 1**: It should be square integrable

$$\Rightarrow \int_{-\infty}^{\infty} |(\psi * \phi)|^2 dx < \infty$$

now,
$$\int_{-\infty}^{\infty} |\psi * \phi|^2 dx = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} \psi(x-u)\phi(u)du\right|^2 dx$$

$$\leq \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |\psi(x-u)||\phi(u)| du\right]^2 dx$$

$$\leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |\psi(x-u)|\,|\phi(u)|^{1/2} du \cdot |\phi(u)|^{1/2}\right)^2 dx \quad -①$$

Acc. to Holder's Inequality:

$$\int fg \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} \quad \text{where } \frac{1}{p}+\frac{1}{q} = 1$$

If $p=q=2$

$$\Rightarrow \left(\int fg\right)^2 \leq \left(\int |f|^2\right) \cdot \left(\int |g|^2\right)$$

Using this in ①

$$\Rightarrow \int_{-\infty}^{\infty} |\psi * \phi|^2 dx \leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |\psi(x-u)|^2 |\phi(u)| du \int_{-\infty}^{\infty} |\phi(u)du\right) dx$$

$$= \int_{-\infty}^{\infty} |\phi(u)du \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-u)|^2 |\phi(u)| du\, dx$$

$$= \int_{-\infty}^{\infty} |\phi(u)du \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-u)|^2 |\phi(u)| dx . du.$$

$$= \int_{-\infty}^{\infty} |\phi(u)|du \cdot \int_{-\infty}^{\infty} |\phi(u)| \left(\int_{-\infty}^{\infty} |\psi(x-u)|^2 dx\right) du$$

As ψ is square integrable

$$\Rightarrow \int_{-\infty}^{\infty} |\psi * \phi|^2 dx \leq \left(\int_{-\infty}^{\infty} |\phi(u)|du\right)^2 < \infty$$ as ϕ is bounded integrable f^n.

$$\Rightarrow \psi * \phi \in L^2(\mathbb{R})$$

Step 2:- Admissibility condition should be satisfied.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{|\widehat{\psi * \phi}(\omega)|^2}{|\omega|} d\omega = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2 |\hat{\phi}(\omega)|^2}{|\omega|} d\omega \leq \sup_{\omega \in R} |\hat{\phi}(\omega)|^2 \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega$$

and $|\hat{\phi}(\omega)| \le \int_{-\infty}^{\infty} |e^{-i\omega x}|\,|\phi(x)|\,dx$

$\hookrightarrow$ exists and finite

$\Rightarrow \psi * \phi$ is bounded.

Parseval's formula :-

$$\left.\begin{aligned} \mathcal{F}(f) &= F \\ \mathcal{F}(g) &= G \end{aligned}\right\} \Rightarrow \langle f, g\rangle = \langle F, G\rangle$$

H.W: $\rightarrow$ Prove it.

- Last class we stated that

$$[\hat{W_\psi} f(a,b)](\omega) = \sqrt{2\pi|a|}\; \hat{f}(\omega)\, \overline{\hat{\psi}(a\omega)}$$

Proof: $[W_\psi f](a,b) = \langle f, \psi_{ab}\rangle = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{\psi}_{ab}(\omega)}\, d\omega = \langle \hat{f}, \hat{\psi}_{ab}\rangle$

(Parseval's formula)

$$= \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega b} \cdot \frac{\sqrt{a}}{\sqrt{2\pi}}\, \overline{\hat{\psi}(a\omega)}\, d\omega$$

$$= \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\cdot e^{+i\omega b}\, \overline{\hat{\psi}(a\omega)}\, d\omega$$

$$= \frac{\sqrt{a}}{\sqrt{2\pi}} \cdot 2\pi \cdot \mathcal{F}^{-1}\left(\hat{f}(\omega)\, \overline{\hat{\psi}(a\omega)}\right)$$

$$\Rightarrow [\hat{W_\psi} f](\omega) = \sqrt{2a|a|}\; \hat{f}(\omega)\, \overline{\hat{\psi}(a\omega)}$$

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Q.1: Find the fourier inversion of

(i) $\dfrac{e^{(2\omega-6)i}}{5-(3-\omega)i}$ (ii) $\dfrac{1+i\omega}{6-\omega^2+5i\omega}$

Ans 1: (i) Let the fourier inversion be f

Ans: (i) $u(t+2)\,e^{-10-(5-3i)t}$

(ii) $u(t)\left[2e^{-3t} - e^{-2t}\right]$

$F(f) = e^{(2\omega-6)i}$

$\Rightarrow$ ~~Inv~~ $f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(2\omega-6)i}\, e^{i\omega x}\, d\omega$

$= \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(2i\omega + ix)\omega}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{(2i+ix)\omega}\cdot e^{-6i}}{5-(3-\omega)i}\, d\omega$

$$\Rightarrow f(x) = \frac{e^{-6i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{(2+x)i\omega}}{5-(3-\omega)i} d\omega$$

Q.2: Find the fourier cosine integral and fourier sine integral of $e^{-x}\cos x$.

Ans 2: $F.C.I = \int_0^\infty A(\omega)\cos\omega x \, d\omega$

where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x \, dx$

$$= \frac{2}{\pi}\int_0^\infty e^{-x}\cos x \cos\omega x \, dx$$

$$= \frac{1}{\pi}\int_0^\infty e^{-x}\left[\cos(x+\omega x) + \cos(x-\omega x) dx\right]$$

$$= \frac{1}{\pi}\int_0^\infty e^{-x}\cos(\omega+1)x \, dx + \frac{1}{\pi}\int_0^\infty e^{-x}\cos(\omega-1)x \, dx$$

$$= \frac{1}{\pi}\left\{ \cancel{\frac{e^{-x}\sin(\omega+1)x}{\omega+1}}\Big|_0^\infty + \int_0^\infty \frac{e^{-x}\sin(\omega+1)x}{\omega+1} dx \right\}$$

$$+ \frac{1}{\pi}\left\{ \frac{e^{-x}\sin(\omega-1)x}{\omega-1}\Big|_0^\infty + \int_0^\infty \frac{e^{-x}\sin(\omega-1)x}{\omega-1} dx \right\}$$

$$= \frac{1}{\pi}\left\{ -e^{-x}\cos(\omega-1 \right.$$

Q.3: Find fourier transform of $f(t) = u(t)e^{-at}$.

Q.4: Use operational formula. $F(\hat{f}(t)) = 2\pi\left[\quad\right]$ to find fourier transform of $\frac{1}{a^2+t^2}$

Ans 4: $\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt = F(\omega)$

$$\Rightarrow \hat{f}(\hat{f}(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t} d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-itx} dx \cdot e^{-i\omega t} dt$$

$$\Rightarrow LHS = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-itx}\,dx\right)e^{-i\omega t}\,dt$$

Doing this we get $\boxed{F(\hat{f}(t)) = 2\pi[f(-\omega)]}$

So to find the fourier transform of $\frac{1}{a^2+t^2}$

<u>given:</u> $\hat{f}(t) = \frac{1}{a^2+\omega^2}$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(t).e^{i\omega x}\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{a^2+\omega^2}\,d\omega$$

Ans: $\frac{\pi}{a}e^{-a|\omega|}$; $a>0$

<u>Checking</u>:

$$f(x) = \frac{\pi}{a}e^{-a|x|}$$

$$\Rightarrow \hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{\pi}{a}.e^{+ax}.e^{-i\omega x}\,dx$$

$$\cancel{= \left(\frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\right)\int_{-\infty}^{\infty}e^{(a+i\omega)x}\,dx}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\pi}{a}.e^{-ax}.e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\int_{-\infty}^{0}e^{(a-i\omega)x}\,dx + \frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\int_{0}^{\infty}e^{-(a+i\omega)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\;\frac{1}{a-i\omega}\,e^{x(a-i\omega)}\Big|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\left(\frac{-1}{a+i\omega}\right).e^{-(a+i\omega)x}\Big|_{0}^{\infty}$$

$$= \left(\frac{1}{\sqrt{2\pi}}.\frac{\pi}{a}\right)\left(\frac{1}{a-i\omega}+\frac{1}{a+i\omega}\right) = \frac{2\pi}{\cancel{a}\sqrt{2\pi}}\left(\frac{\cancel{a}+\cancel{i\omega}+\cancel{a}-\cancel{i\omega}}{a^2+\omega^2}\right)$$

$$= \frac{\sqrt{2\pi}}{a^2+\omega^2}$$

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Results :-

1. $\psi \in L^2(\mathbb{R})$, $C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$

$\Rightarrow \forall f, g \in L^2(\mathbb{R})$, we have

$$\langle W_\psi f, W_\psi g\rangle \leftarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(W_\psi f)(a,b)\}\overline{\{(W_\psi g)(a,b)\}}\, \frac{db.da}{a^2} = C_\psi \langle f, g\rangle$$

This is called Parseval's relation for WT

2. $f \in L^2(\mathbb{R})$

$$\Rightarrow f(x) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b).\ \psi_{a,b}(x)\ \frac{db.da}{a^2}$$ where equality holds almost everywhere.

This is Inversion formula for WT.

Properties of CWT :-

1. $W_\psi(\alpha f + \beta g)(a,b) = \alpha W_\psi(f)(a,b) + \beta W_\psi(g)(a,b)$

2. $[W_\psi f(t-c)](a,b) = (W_\psi f)(a, b-c)$

Proof: LHS : $\int_{-\infty}^{\infty} f(t-c)\overline{\psi_{a,b}(t)}\, dt$ Put $t-c = x$ $\Rightarrow dt = dx$

$$= \int_{-\infty}^{\infty} f(x)\ \overline{\psi_{a,b}(c+x)}\, dx$$

Now $\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$

$$\Rightarrow \psi_{a,b}(c+x) = |a|^{-1/2}\psi\left(\frac{x+c-b}{a}\right) = |a|^{1/2}\psi\left(\frac{x-(b-c)}{a}\right)$$

$$= \psi_{a,b-c}$$

$$\Rightarrow \text{LHS} = \int_{-\infty}^{\infty} f(x)\ \overline{\psi_{a,b-c}}\, dx = (W_\psi f)(a, b-c).$$

3. $(W_\psi D_c f)(a,b) = \frac{1}{\sqrt{c}} W_\psi f\left(\frac{a}{c}, \frac{b}{c}\right)$ where $D_c f(t) = \frac{1}{c} f\left(\frac{t}{c}\right)$; $c > 0$

4. $(W_\psi \phi)(a,b) = \overline{W_\phi \psi\left(\frac{1}{a}, \frac{-b}{a}\right)}$; $a \neq 0$

5. $(W_{\alpha\psi + \beta\phi} f)(a,b) = \bar{\alpha}(W_\psi f)(a,b) + \bar{\beta}(W_\phi f)(a,b)$; $\forall\ \alpha, \beta \in \mathbb{C}$

6. $(W_{P\psi} Pf)(a,b) = (W_\psi f)(a, -b)$ where $Pf(t) = f(-t)$

7. $(W_{T_c \psi} f)(a,b) = (W_\psi f)(a, b + ca)$

8. $(W_{D_c}\psi f)(a,b) = \frac{1}{\sqrt{c}}(W_\psi f)(ac,b)$; $c>0$

H.W: Prove these properties.

→ We say ψ is orthonormal if

$$\psi_{m,n}(x) = 2^{\frac{m}{2}}\psi(2^m x - n), \quad m,n \in \mathbb{Z}$$

is orthonormal, i.e.

$$\langle \psi_{m,n}, \psi_{k,l} \rangle = \delta_{m,k} \cdot \delta_{n,l} = \begin{cases} 1, & \text{if } m=k \text{ and } n=l \\ 0, & \text{otherwise} \end{cases}$$

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→ $\phi, \psi \in L^2(\mathbb{R}) \Rightarrow \langle \psi_{m,k}, \phi_{m,l} \rangle = \langle \psi_{n,k}, \phi_{n,l} \rangle \quad \forall\ m,n,l,k \in \mathbb{Z}$

Here $m \neq n$.

Proof: LHS: $\langle \psi_{m,k}, \phi_{m,l} \rangle = \int_{-\infty}^{\infty} \psi_{m,k}(x)\, \overline{\phi_{m,l}(x)}\, dx$

$$= \int_{-\infty}^{\infty} 2^{\frac{m}{2}}\psi(2^m x - k) \cdot \overline{2^{\frac{m}{2}}\phi(2^m x - l)}\, dx$$

RHS: $\langle \psi_{n,k}, \phi_{n,l} \rangle = \int_{-\infty}^{\infty} 2^{n/2}\psi(2^n x - k) \cdot \overline{2^{n/2}\phi(2^n x - l)}\, dx$

↳ Put $2^n x = 2^m t$

$\Rightarrow 2^n dx = 2^m dt$

$$\Rightarrow \text{RHS: } \int_{-\infty}^{\infty} \cancel{2^n}\psi(2^m t - k) \cdot \overline{\phi(2^m t - l)} \cdot \frac{2^m}{\cancel{2^n}} \cdot dt$$

$$= \int_{-\infty}^{\infty} 2^{m/2}\psi(2^m t - k) \cdot \overline{2^{m/2}(2^m t - l)}\, dt = \text{LHS}.$$

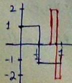
Hence Proved.

→ Let us take ψ as the Haar wavelet. $\psi(x) = \begin{cases} 1 & 0<x<\frac{1}{2} \\ -1 & \frac{1}{2}<x<1 \end{cases}$

we know $\psi_{m,n}(x) = 2^{\frac{m}{2}}\psi(2^m x - n)$

so $\psi_{2,3}(x) = 2\psi(4x-3) = \begin{cases} 2, & \frac{3}{4}<x<\frac{3}{4}+\frac{1}{8} \\ -2, & \frac{3}{4}+\frac{1}{8}<x<1 \end{cases}$

⇒



— Haar wavelet

— $\psi_{2,3}$

$\Rightarrow$ $n>0 \longrightarrow$ +ve translation

$n<0 \longrightarrow$ -ve translation

Translation depends on m also.

$\rightarrow$ $2^m x$ $\longrightarrow$ determines expansion/shrinkage of the width of $\psi_{m,n}$

$m>0 \Rightarrow$ shrinkage

$m<0 \Rightarrow$ expansion.

$\rightarrow$ $2^{m/2}\psi(\;)$ $\longrightarrow$ determines increase/decrease in height.

$m>0 \Rightarrow$ increase in height

$m<0 \Rightarrow$ decrease in height

$$\rightarrow \|\psi_{m,n}(x)\|^2 = \int_{-\infty}^{\infty} \psi_{m,n}(x)\,\overline{\psi_{m,n}(x)}\,dx$$

$$= \int_{-\infty}^{\infty} 2^m \psi(2^m x-n)\,\overline{\psi(2^m x-n)}\,dx$$

$$= \int_{-\infty}^{\infty} 2^m [\psi(2^m x-n)]^2\,dx \quad \text{(Taking } \psi \text{ to be real).}$$

Take $2^m x - n = t$

$\Rightarrow 2^m dx = dt$

$$= \int_{-\infty}^{\infty} [\psi(t)]^2\,dt \qquad \text{(As } \psi \text{ is Haar wavelet).}$$

$$= 1 = \|\psi(x)\|^2$$

$\Rightarrow$ Inner product remains same.

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Last time we saw that $\langle \psi_{m,n}, \psi_{m,n}\rangle = 1$ (as $\|\psi_{m,n}(x)\|^2 = 1$)

<u>Support of a function</u> :- $f : A \subseteq \mathbb{R} \to \mathbb{R}$

$\text{supp } f = \overline{\{x \in A : f(x) \neq 0\}}$

$= \text{closure } \{x \in A : f(x) \neq 0\}$

$=$ set of all <u>limit points</u> of $\{x \in A : f(x) \neq 0\}$

$A \subseteq \mathbb{R}$

$x \in \mathbb{R}$ is called a limit point of A if every interval containing x contains a point of A, other than x.

Proof that $\langle \psi_{m,n}, \psi_{k,l}\rangle$ are orthonormal:

$$\langle \psi_{m,n}, \psi_{k,l}\rangle = \int_{-\infty}^{\infty} 2^{m/2}\psi(2^m x-n)\,.\,2^{k/2}\psi(2^k x-l)\,dx \quad \text{Put } 2^m x - n = t$$

$\Rightarrow 2^m dx = dt$

$$\Rightarrow \langle \psi_{m,n}, \psi_{k,l} \rangle = \int_{-\infty}^{\infty} 2^{\frac{m}{2}+\frac{k}{2}} \cdot \psi(t) \cdot \psi(2^{k} \cdot (n+t) 2^{-m} - l) \, 2^{-m} dt$$

$$= \int_{-\infty}^{\infty} 2^{\frac{k-m}{2}} \cdot \psi(t) \cdot \psi(2^{k-m}(n+t) - l) \, dt \longrightarrow 1$$

When $\underline{m=k}$:

$$\langle \psi_{m,n}, \psi_{m,l} \rangle = \int_{-\infty}^{\infty} \psi(t) \cdot \psi(t+n-l) \, dt$$

$$= \begin{cases} 1 & \text{if } n=l \rightarrow \text{as } \int_{-\infty}^{\infty} (\psi(t))^2 dt = 1 \\ 0 & \text{if } n \neq l \rightarrow \text{as } |n-l| \geq 1 \text{ as } n, l \text{ are integers} \end{cases}$$

hence two haar wavelets concerning only translation will never overlap.

$\Rightarrow \psi(x)$ and $\psi(x+n-l)$ have their supports disjoint from each other

If $m \neq k$:-

a) $\underbrace{k-m}_{=r} > 0$

$$\Rightarrow \langle \psi_{m,n}, \psi_{k,l} \rangle = \int_{-\infty}^{\infty} 2^{r/2} \psi(t) \cdot \psi[2^{r}(n+t) - l] \, dt$$

$$= 2^{r/2} \left\{ \int_{0}^{1/2} \psi[2^{r}(n+t) - l] \, dt - \int_{1/2}^{1} \psi[2^{r}(n+t) - l] \, dt \right\}$$

Put $2^{r}(n+t) - l = p \Rightarrow 2^{r} dt = dp$

$$\Rightarrow \langle \psi_{m,n}, \psi_{k,l} \rangle = 2^{-r/2} \left\{ \int_{2^{r}n-l}^{2^{r}n+2^{r-1}-l} \psi(p) \, dp - \int_{2^{r}n+2^{r-1}-l}^{2^{r}n+2^{r}-l} \psi(p) \, dp \right\}$$

Now: $(2^{r}n + 2^{r-1} - l) - (2^{r}n - l) = 2^{r-1} \geq 1$

and $(2^{r}n + 2^{r} - l) - (2^{r}n + 2^{r-1} - l) = 2^{r} - 2^{r-1} = 2^{r-1}(2-1) = 2^{r-1} \geq 1$

$\rightarrow$ Hence each of the integrals is zero

Hence $\langle \psi_{m,n}, \psi_{k,l} \rangle = 0$ if $m \neq k$

H.W: IIIly for $k < m \rightarrow \langle \psi_{m,n}, \psi_{k,l} \rangle = 0$

→ $\psi(t) = (1-t^2)e^{-at^2/2}$

For $a=1$, draw the graph of $\psi(t)$.

Ans: $\psi(t) = (1-t^2)e^{-t^2/2}$.

At $t=0$: $\psi(0)=1$; $\psi(t)=0$ for $t=\pm 1$

$t \to \infty$; $\psi(t) \to 0$

$t \to -\infty$; $\psi(t) \to 0$

$$\frac{d\psi}{dt} = -2te^{-t^2/2} + (1-t^2)(-t)e^{-t^2/2} = 0$$

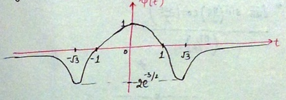
$$\Rightarrow (-te^{-t^2/2})(2+1-t^2) = 0$$

$$\Rightarrow t=0,\ t=\sqrt{3},\ t=-\sqrt{3}.$$

$$\frac{d^2\psi}{dt^2} = -2e^{-t^2/2} - 2t(-t)e^{-t^2/2} + (1-t^2)(-1)e^{-t^2/2} + (2t)(-t)e^{-t^2/2} + (1-t^2)(-t)(-t)e^{-t^2/2}$$

$$\psi(\sqrt{3}) = \psi(-\sqrt{3}) = -2e^{-3/2}$$

So the graph of the function is:



Here $\psi \in L^2(\mathbb{R})$

H.W: Check that $C_\psi < \infty$

Hence this is a wavelet.

This is called Mexican Hat wavelet.

H.W: Check that $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$ is orthonormal.

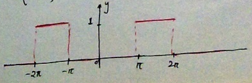
H.W: $f: \mathbb{R} \to \mathbb{R}$ is a wavelet. f is periodic? – True/False

Shannon's Wavelet :-

This is a function whose fourier transform is

$\chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(t)$ → This is characteristic function.

$$= \begin{cases} 1, & \text{if } t \in [-2\pi,-\pi]\cup[\pi,2\pi] \\ 0, & \text{otherwise} \end{cases}$$



$\Rightarrow$ Shannon wavelet is inverse fourier transform of χ.

$$\Rightarrow \psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \hat{\psi}(t)\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-2\pi}^{-\pi} e^{i\omega t} d\omega + \int_{\pi}^{2\pi} e^{i\omega t} d\omega \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{(e^{-i\omega\pi} - e^{i\omega 2\pi})}{it} + \frac{e^{i\omega 2\pi} - e^{i\omega\pi}}{it} \right\}$$

$$= \frac{1}{it(\sqrt{2\pi})} \left\{ \cancel{\cos \omega\pi} - i\sin\omega\pi - \cancel{\cos(2\omega\pi)} + i\sin(2\omega\pi) + \cancel{\cos(2\omega\pi)} + i\sin 2\pi\omega - \cancel{\cos\omega\pi} - i\sin\omega\pi \right\}$$

$$= \frac{1}{\cancel{i}t(\sqrt{2\pi})} \cdot 2\cancel{i} \left\{ \sin 2\pi t - \sin \pi t \right\}$$

$$= \frac{\sqrt{2\pi}\, \sin\left(\frac{\pi t}{2}\right) \cos\left(\frac{3\pi t}{2}\right)}{\left(\pi t/2\right)}$$

$\rule{1cm}{0.4pt}\times\rule{1cm}{0.4pt}$

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$\underset{\sim}{Z\text{- transform}}$

Given $\{a_n\}$

$$f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C} \ \forall x \in \mathbb{C}$$

↳ This is called a power series.

The set of all $x \in \mathbb{C}$ such that the power series converges is called the region of convergence of the power series. This region of convergence is a domain of the function defined in terms of the above Power series.

Definition of Z-transform :-

Given $\{f_n\}$ a sequence of complex/real numbers,

(**Note** : A real sequence is the range of a function from $\mathbb{N}$ to $\mathbb{R}$)

$$\boxed{Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n}}$$

Result: Given $\sum_{n \geq 0} a_n w^n$

If it is convergent for $w = w_0 \neq 0$ then it is convergent for all w in $\{w : |w| < |w_0|\}$

Proof: $\sum_{n \geq 0} a_n w^n = \sum_{n \geq 0} a_n w_0^n \cdot \frac{w^n}{w_0^n} < \infty$ if $\left|\frac{w}{w_0}\right| < 1 \Rightarrow |w| < |w_0|$

Hence Proved.

Q.1: $f_0 = 1$, $f_n = 0 \ \forall n$. Find $Z\{f_n\}$

Ans.1: $Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} = f_0 z^{-0} = 1$ ROC $= \mathbb{C}$

Q.2: $f_n = 1 \ \forall n \geq 0$. Find $Z\{f_n\}$.

Ans.2: $Z\{f_n\} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \frac{1}{1 - \frac{1}{z}}$ $\left|\frac{1}{z}\right| < 1$

$= \frac{z}{z-1}$ ROC : $|z| > 1$.

Q.3: $f_n = \frac{a^n}{n!}$ Find $Z\{f_n\}$.

Ans.3: $f_n = \frac{a^n}{n!}$; $Z\{f_n\} = \sum_{n=0}^{\infty} \frac{a^n \cdot z^{-n}}{n!} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \cdot \frac{1}{n!} = e^{a/z}$; ROC : $|z| >$ [illegible]

Q.4: $f_n = e^{n^3}$. Find $Z\{f_n\}$

Ans.4: $Z\{f_n\} = \sum_{n=0}^{\infty} e^{n^3} \cdot z^{-n}$ — Doesn't exist.

Theorem :- $\{f_n\}$ is of exponential type, i.e. $\exists$ $N > 0$, $t_0 \geq 0$ and $n_0 \geq 0$

such that $|f_n| < Ne^{nt_0}$ $\forall\ n \geq n_0$

$\Leftrightarrow Z\{f_n\}$ exists.

H.W: Prove this.

Linearity: $Z\{\alpha f_n + \beta g_n\} = \alpha Z\{f_n\} + \beta Z\{g_n\}$ $\forall\ \alpha, \beta \in \mathbb{C}$

Q.5: Find $Z\{\sin h n\theta\}$

Ans.5: $f_n = \sinh n\theta = \frac{e^{in\theta}}{2} - \frac{e^{-in\theta}}{2}$

$$\Rightarrow Z\{f_n\} = \frac{1}{2}\left\{\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{e^{-i\theta}}{z}\right)^n\right\}$$

ROC: $\left|\frac{e^{i\theta}}{z}\right| < 1$ & $\left|\frac{e^{-\theta}}{z}\right| < 1$ $\Rightarrow |z| > e^{\theta}$ $\Rightarrow \boxed{|z| > e^{\theta}}$

$$= \frac{1}{2}\left\{\frac{1}{1-\frac{e^{i\theta}}{z}} + \frac{1}{1-\frac{e^{-i\theta}}{z}}\right\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-e^{i\theta}} + \frac{z}{z-e^{-i\theta}}\right\}$$

$$= \frac{1}{2}\left\{\frac{z^2 - ze^{-i\theta} + z^2 - ze^{i\theta}}{z^2 - z\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)2 + 1}\right\}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\left\{\frac{z^2 - z\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}{(\quad)}\right\}$$

$$= \frac{z^2 - z\cosh n\theta}{z^2 - 2z\cosh n\theta + 1}$$

ROC: $|z| > e^{\theta}$

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Q.1: Find $Z\{\sin n\}$.

Ans.1: $f_n = \sin n = \frac{e^{in} - e^{-in}}{2i} \Rightarrow |f_n| = \left|\frac{e^{in} - e^{-in}}{2i}\right| < \left|\frac{e^{in}}{2i}\right| + \left|\frac{e^{in}}{2i}\right| = \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow |f_n| < 1 < Ne^{nt_0}$ $\forall\ n \geq n_0$

Hence $Z\{f_n\}$ exists.

$$\Rightarrow Z\{\sin n\} = \sum_{n=0}^{\infty}\frac{e^{in} - e^{in}}{2i}\cdot z^{-n} = \frac{1}{2i}\left\{\sum_{n=0}^{\infty}\left(\frac{e^i}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{e^i z}\right)^n\right\}$$

ROC: $\left|\frac{e^i}{z}\right| < 1 \Rightarrow \boxed{|z| > 1}$

$$= \frac{1}{2i}\left\{\frac{1}{1-\frac{e^i}{z}} - \frac{1}{1-\frac{e^{-i}}{z}}\right\}$$

$$= \frac{1}{2i}\left\{\frac{z}{z-e^i} - \frac{z}{z-e^{-i}}\right\} = \frac{1}{2i}\left\{\frac{\cancel{z^2} - ze^{-i} - \cancel{z^2} + ze^{i}}{z^2 - z(e^i + e^{-i}) + 1}\right\}$$

$$= \frac{z\sin 1}{z^2 - 2z\cos 1 + 1}$$

Theorem : (Shifting)

Let $F(z) = Z\{f_n\}$, Valid of $|z| > \frac{1}{R}$

Then (a) $Z\{f_{n-k}\} = z^{-k} F(z)$

(b) $Z\{f_{n+k}\} = z^{k}\left[F(z) - \sum_{n=0}^{k-1} f_n z^{-n}\right]$

→ If $\{f_n\} \equiv f_0, f_1, f_2, \dots f_k, f_{k+1}, \dots$

$\{f_{n-k}\} \equiv f_{-k}, f_{1-k}, \dots, f_0, f_1, \dots$

$\equiv 0, 0, \dots, f_0, f_1, \dots$

$\{f_{n+k}\} \equiv f_k, f_{k+1}, \dots$

Proof :

a) $Z\{f_{n-k}\} = Z\{0, 0, 0 \dots, f_0, f_1, \dots\}$

$= \sum_{n \geq 0} f_{n-k} z^{-n}$

$= f_{-k} + f_{1-k} z^{-1} + f_{2-k} z^{-2} + \dots + z^{-k} f_0 + z^{-k-1} f_1 + z^{-k-2} f_2 + \dots$

$= 0 + 0 + 0 + \dots + z^{-k}\{f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots\}$

$= z^{-k} \sum_{n \geq 0} f_n z^{-n}$

$= z^{-k} F(z)$ //

b) $Z\{f_{n+k}\} = \sum_{n \geq 0} f_{n+k} z^{-n}$

$= f_k + f_{k+1} z^{-1} + f_{k+2} z^{-2} + \dots$

$= \frac{1}{z^{-k}}\{f_k z^{-k} + f_{k+1} z^{-k-1} + f_{k+2} z^{-k-2} + \dots\}$

$= z^{k}\{\underline{f_0} + \underline{f_1 z^{-1}} - f_1 z^{-1} - f_0 + \underline{f_2 z^{-2}} - f_2 z^{-2} + \dots + \underline{f_{k-1} z^{-k+1}} - f_{k-1} z^{-k+1}$
$+ \underline{f_k z^{-k}} + \underline{f_{k+1} z^{-k-1}} + \dots\}$

$= z^{k}\{F(z) - f_0 - f_1 z^{-1} - f_2 z^{-2} \dots - f_{k-1} z^{-(k-1)}\}$

$= z^{k}\{F(z) - \sum_{n=0}^{k-1} f_n z^{-n}\}$ //

Q.2: Find $Z\{\cos(n+1)\theta\}$

Theorem: (Scaling)

Let $Z\{f_n\} = F(z)$, $|z| > \frac{1}{R}$ and $a \in \mathbb{C}$

$\Rightarrow Z\{a^{-n} f_n\} = F(az)$

Proof: Given $\sum_{n \geq 0} f_n z^{-n} = F(z)$

$$\Rightarrow Z\{a^{-n} f_n\} = \sum_{n \geq 0} a^{-n} f_n z^{-n} = \sum_{n \geq 0} f_n (az)^{-n} = F(az)$$

Q.3: Find $Z\{e^{-n} \sin 2n\}$

Theorem :-

Let $Z\{f_n\} = F(z)$, $|z| > \frac{1}{R}$

$$\Rightarrow f_p = \lim_{z \to \infty} \left[z^p \left\{ F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2} - \cdots - \frac{f_{p-1}}{z^{p-1}} \right\} \right]$$

$p = 0, 1, 2, \ldots$

Proof: $F(z) = \sum_{n \geq 0} f_n z^{-n}$

$$\Rightarrow z^p F(z) = \sum_{n \geq 0} f_n z^{-n} \cdot z^p$$

$$= f_0 z^p + z^{p-1} f_1 + \cdots + f_p + \frac{f_{p+1}}{z} + \cdots$$

$$\Rightarrow z^p \left\{ F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2} - \cdots - \frac{f_{p-1}}{z^{p-1}} \right\}$$

$$= \cancel{f_0 z^p} + \cancel{z^{p-1} f_1} + \cancel{\cdots} + f_p + \frac{f_{p+1}}{z} + \cdots$$

$$- \cancel{f_0 z^p} - \cancel{z^{p-1} f_1} - \cancel{z^{p-2} f_2} - \cdots - \cancel{z f_{p-1}}$$

$$= f_p + \frac{f_{p+1}}{z} + \frac{f_{p+2}}{z^2} + \cdots$$

$$\Rightarrow \lim_{z \to \infty} z^p \left\{ F(z) - f_0 - \frac{f_1}{z} - \cdots - \frac{f_{p-1}}{z^{p-1}} \right\} = f_p$$

Q.4: If $Z\{f_n\} = F(z) = \frac{3z^2 - 4z + 7}{(z-1)^3}$ then find f_0, f_1 and f_2.

Ans.4: $f_0 = \lim_{z \to \infty} F(z) = 0$

$f_1 = \lim_{z \to \infty} z\{F(z)\} = \lim_{z \to \infty} \frac{3z^3 - 4z^2 + 7z}{(z-1)^3} = 3$

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Recall:

$f: \mathbb{R} \to \mathbb{R}$ periodic $p = 2L$

piecewise cts.

$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n x \pi}{L} + b_n \sin \frac{n x \pi}{L} \right)$

where $a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n x \pi}{L}\,dx$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n x \pi}{L}\,dx \qquad n = 1, 2, 3 \ldots$

Also $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}}$

$= c_0 + \sum_{n=1}^{\infty} c_n \left[\cos (n \omega_0 x + \delta_n) \right]$ where $\omega_0 = \frac{\pi}{L}$

$c_n = \sqrt{a_n^2 + b_n^2}$

$\delta = \tan^{-1}\left(-b_n/a_n\right)$

This form of fourier series is called

Phase Angle Form.

Convolution of two sequences:-

Let $\{f_n\} = f_0, f_1, f_2, \ldots$

$\{g_n\} = g_0, g_1, g_2, \ldots$

$\{f_n\} * \{g_n\}$ is a sequence whose nth term is

$\{f_n * g_n\} = f_0 g_n + f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0 = \sum_{k=0}^{n} f_k g_{n-k}$

Q1: $f_n = n$; $g_n = \frac{1}{n}$, $n = 1, 2, 3 \ldots$

Find $f_n * g_n$.

Properties of convolution:

1) $\{f_n * g_n\} = \{g_n * f_n\}$ — commutative

2) $\{(u_n + v_n) * f_n\} = \{u_n * f_n\} + \{v_n * f_n\}$ — distributive.

3) $\{u_n * v_n\} * \{g_n\} = \{u_n\} * \{v_n * g_n\}$ — associative

Theorem: $Z\{f_n\} = F(z)$

$Z\{g_n\} = G(z) \Rightarrow Z\{f_n * g_n\} = F(z) \cdot G(z)$

Q.2: verify $\left\{\frac{1}{n!}\right\} * \left\{\frac{1}{n!}\right\} = \left\{\frac{2^n}{n!}\right\}$

Q.3: $f_n = p^n$; $g_n = q^n$, Verify the convolution theorem for Z-transforms.

Difference Equation:

Let $n, n+1, \dots, n+k$ be $(k+1)$ positive integers

$y_n, y_{n+1}, \dots y_{n+k}$ be values of a function,

then an equation of the form:

$$y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+k-2} + \dots + y_n a_k = f(n)$$

is called difference equation.

The a_i's are known to us and the y_i's are to be found out.

e.g. $y_{n+1} - 5y_n = 0$ is a difference equation.

We can solve it by Z-transform.

Q.4: Solve $y_{n+1} - 5y_n = 0$ by Z-transform

Ans-4: $y_{n+1} - 5y_n = 0$

$\Rightarrow$ Applying Z-transform both sides:

$$Z\{y_{n+1}\} - 5Z\{y_n\} = Z(0)$$

$$\Rightarrow z(F(z) - y_0) - 5F(z) = 0$$

$$\Rightarrow zF(z) - 5F(z) = zy_0$$

$$\Rightarrow F(z) = \frac{zy_0}{z-5} = \frac{y_0}{1-5z^{-1}} = (5)^n y_0 u(n) = 5^n y_0$$

Q.5: Find $Z\{4n^2 + n + 2\}$

Result: $Z\{nf_n\} = -z\frac{d}{dz}[F(z)]$

in general, $Z\{n^p f_n\} = -z\frac{d}{dz}\left[Z\{n^{p-1} f_n\}\right]$