An integrating factor is $e^{\int(1/6)\,dt} = e^{t/6}$. Multiply the differential equation by this factor to obtain

$$v\,e^{t/6} + \frac{1}{6}e^{t/6}v = \left(ve^{t/6}\right) = 4e^{t/6}$$

and integrate to get

$$ve^{t/6} = 24e^{t/6} + C.$$

The velocity is

$$v(t) = 24 + Ce^{-t/6}.$$

Since the block starts from rest at time zero, $v(0) = 0 = 24 + C$, so $C = -24$ and

$$v(t) = 24\left(1 - e^{-t/6}\right)$$

Let $x(t)$ be the position of the block at any time, measured from the top of the plane. Since $v(t) = x'(t)$, we get

$$x(t) = \int v(t)\,dt = 24t + 144e^{-t/6} + K.$$

If we let the top of the block be the origin along the inclined plane, then $x(0) = 0 = 144 + K$, so

$$K = -144.$$

The position function is

$$x(t) = 24t + 144\left(e^{-t/6} - 1\right).$$

We can now determine the block's position and velocity at any time.

Suppose, for example, we want to know when the block reaches the bottom of the ramp. This happens when the block has gone 50 feet. If this occurs at time T, then

$$x(T) = 50 = 24T + 144\left(e^{-T/6} - 1\right)$$

This transcendental equation cannot be solved algebraically for T, but a computer approximation yields $T \approx 5.8$ seconds.

Notice that

$$\lim_{t\to\infty} v(t) = 24,$$

which means that the block sliding down the ramp has a terminal velocity. If the ramp is long enough, the block will eventually settle into a slide of approximately constant velocity.

The mathematical model we have constructed for the sliding block can be used to analyze the motion of the block under a variety of conditions. For example, we can solve the equations leaving θ arbitrary, and determine the influence of the slope angle of the ramp on position and velocity. Or we could leave μ unspecified and study the influence of friction on the motion.

1.7.2 Electrical Circuits

Electrical engineers often use differential equations to model circuits. The mathematical model is used to analyze the behavior of circuits under various conditions, and aids in the design of circuits having specific characteristics.

We will look at simple circuits having only resistors, inductors and capacitors. A capacitor is a storage device consisting of two plates of conducting material isolated from one another by an insulating material, or dielectric. Electrons can be transferred from one plate to another via external circuitry by applying an electromotive force to the circuit. The charge on a capacitor is essentially a count of the difference between the numbers of electrons on the two plates. This charge is proportional to the applied electromotive force, and the constant of proportionality is the capacitance. Capacitance is usually a very small number, given in micro (10^{-6}) or pi (10^{-12}) farads. To simplify examples and problems, some of the capacitors in this book a assigned numerical values that would actually make them occupy large buildings.

An inductor is made by winding a conductor such as wire around a core of magnetic ma rial. When a current is passed through the wire, a magnetic field is created in the core and arou the inductor. The voltage drop across an inductor is proportional to the change in the curre flow, and this constant of proportionality is the inductance of the inductor, measured in henr

Current is measured in amperes, with one amp equivalent to a rate of electron flow of o coulomb per second. Charge $q(t)$ and current $i(t)$ are related by

$$i(t) = q'(t).$$

The voltage drop across a resistor having resistance R is iR. The drop across a capacitor havi capacitance C is q/C. And the voltage drop across an inductor having inductance L is $Li'(t)$.

We construct equations for a circuit by using Kirchhoff's current and voltage la Kirchhoff's current law states that the algebraic sum of the currents at any juncture of circuit is zero. This means that the total current entering the junction must balance the curre leaving (conservation of energy). Kirchhoff's voltage law states that the algebraic sum of t potential rises and drops around any closed loop in a circuit is zero.

As an example of modeling a circuit mathematically, consider the circuit of Figure 1. Starting at point A, move clockwise around the circuit, first crossing the battery, where the is an increase in potential of E volts. Next there is a decrease in potential of iR volts acr the resistor. Finally, there is a decrease of $Li'(t)$ across the inductor, after which we return point A. By Kirchhoff's voltage law,

$$E - iR - Li' = 0,$$

which is the linear equation

$$i' + \frac{R}{L}i = \frac{E}{L}$$

Solve this to obtain

$$i(t) = \frac{E}{R} + Ke^{-Rt/L}$$

To determine the constant K, we need to be given the current at some time. Even without th we can tell from this equation that as $t \to \infty$, the current approaches the limiting value E This is the steady-state value of the current in the circuit.

Another way to derive the differential equation of this circuit is to designate one of the components as a source, then set the voltage drop across that component equal to the sum of the voltage drops across the other components. To see this approach, consider the circuit in Figure 1.18. Suppose the switch is initially open so that no current flows, and that the char

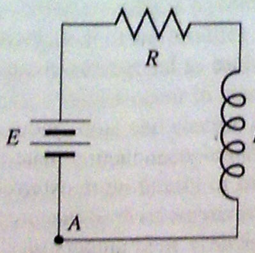

FIGURE 1.17 *RL Circuit.*

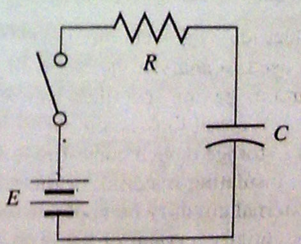

FIGURE 1.18 *RC circuit.*

on the capacitor is zero. At time zero, close the switch. We want the charge on the capacitor. Notice that we have to close the switch before there is a loop. Using the battery as a source, write

$$iR + \frac{1}{C}q = E,$$

or

$$Rq' + \frac{1}{C}q = E.$$

This leads to the linear equation

$$q' + \frac{1}{RC}q = \frac{E}{R},$$

with solution

$$q(t) = EC\left(1 - e^{-t/RC}\right)$$

satisfying $q(0) = 0$. This equation provides a good deal information about the circuit. Since the voltage on the capacitor at time t is $q(t)/C$, or $E(1 - e^{-t/RC})$, we can see that the voltage approaches E as $t \to \infty$. Since E is the battery potential, the difference between battery and capacitor voltages becomes negligible as time increases, indicating a very small voltage drop across the resistor.

The current in this circuit can be computed as

$$i(t) = q'(t) = \frac{E}{R}e^{-t/RC}$$

after the circuit is switched on. Thus $i(t) \to E/R$ as $t \to \infty$.

Often we encounter discontinuous currents and potential functions in dealing with circuits. These can be treated using Laplace transform techniques, which we will discuss in Chapter 3.

1.7.3 Orthogonal Trajectories

Two curves intersecting at a point P are said to be *orthogonal* if their tangents are perpendicular (orthogonal) at P. Two families of curves, or trajectories, are orthogonal if each curve of the first family is orthogonal to each curve of the second family, wherever an intersection occurs. Orthogonal families occur in many contexts. Parallels and meridians on a globe are orthogonal, as are equipotential and electric lines of force.

A problem that occupied Newton and other early developers of the calculus was the determination of the family of orthogonal trajectories of a given family of curves. Suppose we are given a family $\mathfrak{F}$ of curves in the plane. We want to construct a second family $\mathfrak{G}$ of curves so that every curve in $\mathfrak{F}$ is orthogonal to every curve in $\mathfrak{G}$ wherever an intersection occurs. As a simple example, suppose $\mathfrak{F}$ consists of all circles about the origin. Then $\mathfrak{G}$ consists of all straight lines through the origin (Figure 1.19). It is clear that each straight line is orthogonal to each circle wherever the two intersect.

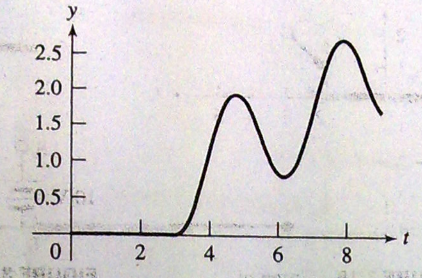

FIGURE 3.17 *Solution of*

$$y'' + 4y = \begin{cases} 0 & \text{if } 0 \leq t < 3 \\ t & \text{if } t \leq 3 \end{cases} ; \quad (y(0) = y'(0) = 0).$$

Often we need to write a function having several jump discontinuities in terms of Heaviside functions in order to use the shifting theorems. Here is an example.

EXAMPLE 3.14

Let

$$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ t-1 & \text{if } 2 \leq t < 3 \\ -4 & \text{if } t \geq 3. \end{cases}$$

A graph of f is shown in Figure 3.18. There are jump discontinuities of magnitude 1 at $t = 2$ and magnitude 6 at $t = 3$.

Think of $f(t)$ as consisting of two nonzero parts, the part that is $t - 1$ on $[2, 3)$ and the part that is -4 on $[3, \infty)$. We want to turn on $t - 1$ at time 2 and turn it off at time 3, then turn on -4 at time 3 and leave it on.

The first effect is achieved by multiplying the pulse function $H(t-2) - H(t-3)$ by $t - 1$. The second is achieved by multiplying $H(t-3)$ by 4. Therefore

$$f(t) = [H(t-2) - H(t-3)](t-1) - 4H(t-3).$$

As a check, this gives $f(t) = 0$ if $t < 2$ because all of the shifted Heaviside functions are zero for $t < 2$. For $2 \leq t < 3$, $H(t-2) = 1$ but $H(t-3) = 0$ so $f(t) = t - 1$. And for $t \geq 3$, $H(t-2) = H(t-3) = 1$, so $f(t) = -4$. ∎

3.3.4 Analysis of Electrical Circuits

The Heaviside function is important in many kinds of problems, including the analysis of electrical circuits, where we anticipate turning switches on and off. Here are two examples.

EXAMPLE 3.15

Suppose the capacitor in the circuit of Figure 3.19 initially has zero charge and that there is no initial current. At time $t = 2$ seconds, the switch is thrown from position B to A, held there for 1 second, then switched back to B. We want the output voltage E_{out} on the capacitor.

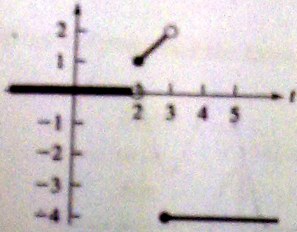

FIGURE 3.18 Graph of

$$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ t-1 & \text{if } 2 \le t < 3. \\ -4 & \text{if } t \ge 3 \end{cases}$$

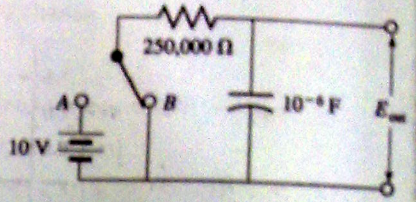

FIGURE 3.19

From the circuit diagram, the forcing function is zero until $t = 2$, then has value 10 until $t = 3$, and then is zero again. Thus E is the pulse function

$$E(t) = 10[H(t-2) - H(t-3)].$$

By Kirchhoff's voltage law,

$$Ri(t) + \frac{1}{C}q(t) = E(t),$$

or

$$250,000q'(t) + 10^6 q(t) = E(t).$$

We want to solve for q subject to the initial condition $q(0) = 0$. Apply the Laplace transform to the differential equation, incorporating the initial condition, to write

$$250,000[sQ(t) - q(0)] + 10^6 Q(t) = 250,000sQ + 10^6 Q = \mathfrak{L}[E(t)].$$

Now

$$\mathfrak{L}[E(t)](s) = 10\mathfrak{L}[H(t-2)](s) - 10\mathfrak{L}[H(t-3)](s)$$
$$= \frac{10}{s}e^{-2s} - \frac{10}{s}e^{-3s}.$$

We now have the following equation for Q:

$$2.5(10^5)sQ(s) + 10^6 Q(s) = \frac{10}{s}e^{-2s} - \frac{10}{s}e^{-3s}$$

or

$$Q(s) = 4(10^{-5})\frac{1}{s(s+4)}e^{-2s} - 4(10)^{-5}\frac{1}{s(s+4)}e^{-3s}.$$

Use a partial fractions decomposition to write

$$Q(s) = 10^{-5}\left[\frac{1}{s}e^{-2s} - \frac{1}{s+4}e^{-2s}\right] - 10^{-5}\left[\frac{1}{s}e^{-3s} - \frac{1}{s+4}e^{-3s}\right].$$

By the second shifting theorem,

$$\mathfrak{L}^{-1}\left[\frac{1}{s}e^{-2s}\right](t) = H(t-2)$$

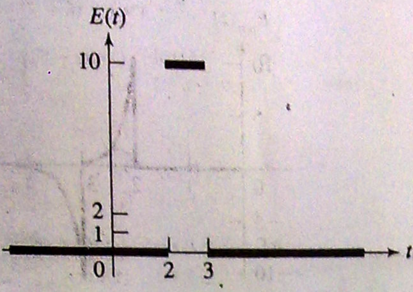

FIGURE 3.20 *Input voltage for the circuit of Figure 3.19.*

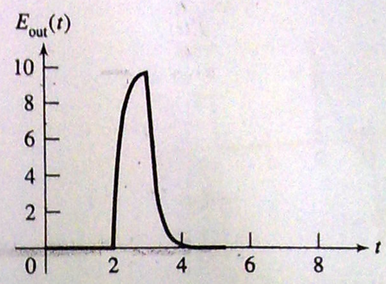

FIGURE 3.21 *Output voltage for the circuit of Figure 3.19.*

and

$$\mathfrak{L}^{-1}\left[\frac{1}{s+4}e^{-2s}\right] = H(t-2)f(t-2),$$

where $f(t) = \mathfrak{L}^{-1}[1/(s+4)] = e^{-4t}$. Thus

$$\mathfrak{L}^{-1}\left[\frac{1}{s+4}e^{-2s}\right] = H(t-2)e^{-4(t-2)}.$$

The other two terms in $Q(s)$ are treated similarly, and we obtain

$$q(t) = 10^{-5}[H(t-2) - H(t-2)e^{-4(t-2)}] - 10^{-5}[H(t-3) - H(t-3)e^{-4(t-3)}]$$

$$= 10^{-5}H(t-2)[1-e^{-4(t-2)}] - 10^{-5}H(t-3)[1-e^{-4(t-3)}].$$

Finally, since the output voltage is $E_{\text{out}}(t) = 10^6 q(t)$,

$$E_{\text{out}}(t) = 10H(t-2)[1-e^{-4(t-2)}] - 10H(t-3)[1-e^{-4(t-3)}].$$

The input and output voltages are graphed in Figures 3.20 and 3.21. ■

EXAMPLE 3.16

The circuit of Figure 3.22 has the roles of resistor and capacitor interchanged from the circuit of the preceding example. We want to know the output voltage $i(t)R$ at any time.

The differential equation of the preceding example applies to this circuit, but now we are interested in the current. Since $i = q'$, then

$$(2.5)(10^5)i(t) + 10^6 q(t) = E(t); \qquad i(0) = q(0) = 0.$$

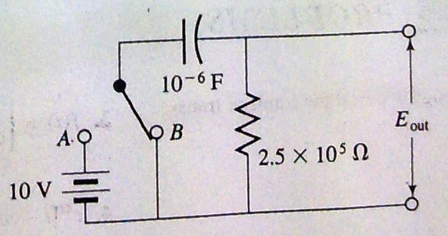

FIGURE 3.22

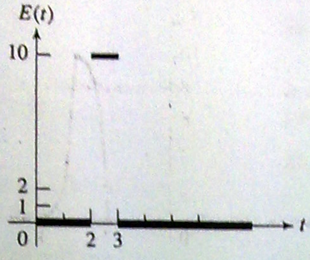

FIGURE 3.23 *Input voltage for the circuit of Figure 3.22.*

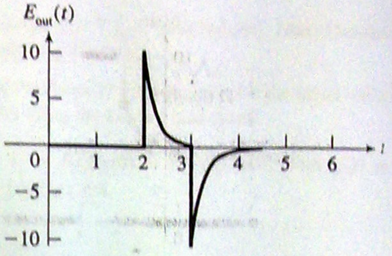

FIGURE 3.24 *Output voltage for the circuit of Figure 3.22.*

The strategy of eliminating q by differentiating and using $i = q'$ does not apply because $E(t)$ is not differentiable. To eliminate $q(t)$ in the present case, write

$$q(t) = \int_0^t i(\tau)\,d\tau + q(0) = \int_0^t i(\tau)\,d\tau.$$

We now have the following problem to solve for the current:

$$(2.5)(10^5)i(t) + 10^6 \int_0^t i(\tau)\,d\tau = E(t); \qquad i(0) = 0.$$

This is not a differential equation. Nevertheless, we have the means to solve it. Take the Laplace transform of the equation, using equation (3.4), to obtain

$$(2.5)(10^5)I(s) + 10^6 \frac{1}{s} I(s) = \mathfrak{L}[E](s)$$
$$= 10\frac{1}{s}e^{-2s} - 10\frac{1}{s}e^{-3s}.$$

Here $I = \mathfrak{L}[i]$. Solve for $I(s)$ to get

$$I(s) = 4(10^{-5})\frac{1}{s+4}e^{-2s} - 4(10^{-5})\frac{1}{s+4}e^{-3s}.$$

Take the inverse Laplace transform to obtain

$$i(t) = 4(10^{-5})H(t-2)e^{-4(t-2)} - 4(10^{-5})H(t-3)e^{-4(t-3)}$$

The input and output voltages are graphed in Figures 3.23 and 3.24. ■

PROBLEMS

In each of Problems 1 through 15, find the Laplace transform of the function.

1. $(t^3 - 3t + 2)e^{-2t}$
2. $e^{-3t}(t-2)$
3. $f(t) = \begin{cases} 1 & \text{for } 0 \le t < 7 \\ \cos(t) & \text{for } t \ge 7 \end{cases}$
4. $e^{4t}[t - \cos(t)]$

5. $f(t) = \begin{cases} t & \text{for } 0 \le t < 3 \\ 1-3t & \text{for } t \ge 3 \end{cases}$

6. $f(t) = \begin{cases} 2t - \sin(t) & \text{for } 0 \le t < \pi \\ 0 & \text{for } t \ge \pi \end{cases}$

7. $e^{-t}[1 - t^2 + \sin(t)]$

8. $f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2 \\ 1 - t - 3t^2 & \text{for } t \ge 2 \end{cases}$

9. $f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi \\ 2 - \sin(t) & \text{for } t \ge 2\pi \end{cases}$

10. $f(t) = \begin{cases} -4 & \text{for } 0 \le t < 1 \\ 0 & \text{for } 1 \le t < 3 \\ e^{-t} & \text{for } t \ge 3 \end{cases}$

11. $te^{-2t}\cos(3t)$

12. $e^{t}[1 - \cosh(t)]$

13. $f(t) = \begin{cases} t-2 & \text{for } 0 \le t < 16 \\ -1 & \text{for } t \ge 16 \end{cases}$

14. $f(t) = \begin{cases} 1 - \cos(2t) & \text{for } 0 \le t < 3\pi \\ 0 & \text{for } t \ge 3\pi \end{cases}$

15. $e^{-5t}(t^4 + 2t^2 + t)$

In each of Problems 16 through 25, find the inverse Laplace transform of the function.

16. $\dfrac{1}{s^2+4s+12}$

17. $\dfrac{1}{s^2-4s+5}$

18. $\dfrac{1}{s^3}e^{-5s}$

19. $\dfrac{se^{-2s}}{s^2+9}$

20. $\dfrac{3}{s+2}e^{-4s}$

21. $\dfrac{1}{s^2+6s+7}$

22. $\dfrac{s-4}{s^2-8s+10}$

23. $\dfrac{s+2}{s^2+6s+1}$

24. $\dfrac{1}{(s-5)^3}e^{-s}$

25. $\dfrac{1}{s(s^2+16)}e^{-21s}$

26. Determine $\mathfrak{L}[e^{-2t}\int_0^t e^{2w}\cos(3w)\,dw]$. *Hint:* Use the first shifting theorem.

In each of Problems 27 through 32, solve the initial value problem by using the Laplace transform.

27. $y'' + 4y = f(t)$; $y(0) = 1, y'(0) = 0$, with $f(t) = \begin{cases} 0 & \text{for } 0 \le t < 4 \\ 3 & \text{for } t \ge 4 \end{cases}$

28. $y'' - 2y' - 3y = f(t)$; $y(0) = 1, y'(0) = 0$, with $f(t) = \begin{cases} 0 & \text{for } 0 \le t < 4 \\ 12 & \text{for } t \ge 4 \end{cases}$

29. $y^{(3)} - 8y = g(t)$; $y(0) = y'(0) = y''(0) = 0$, with $g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 2 & \text{for } t \ge 6 \end{cases}$

30. $y'' + 5y' + 6y = f(t)$; $y(0) = y'(0) = 0$, with $f(t) = \begin{cases} -2 & \text{for } 0 \le t < 3 \\ 0 & \text{for } t \ge 3 \end{cases}$

31. $y^{(3)} - y'' + 4y' - 4y = f(t)$; $y(0) = y'(0) = 0$, $y''(0) = 1$, with $f(t) = \begin{cases} 1 & \text{for } 0 \le t < 5 \\ 2 & \text{for } t \ge 5 \end{cases}$

32. $y'' - 4y' + 4y = f(t)$; $y(0) = -2, y'(0) = 1$, with $f(t) = \begin{cases} t & \text{for } 0 \le t < 3 \\ t+2 & \text{for } t \ge 3 \end{cases}$

33. Calculate and graph the output voltage in the circuit of Figure 3.19, assuming that at time zero the capacitor is charged to a potential of 5 volts and the switch is opened at 0 and closed 5 seconds later.

34. Calculate and graph the output voltage in the RL circuit of Figure 3.25 if the current is initially zero and $E(t) = \begin{cases} 0 & \text{for } 0 \le t < 5 \\ 2 & \text{for } t \ge 5. \end{cases}$

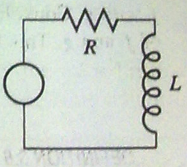

FIGURE 3.25

35. Solve for the current in the RL circuit of Problem 34 if the current is initially zero and $E(t) = \begin{cases} k & \text{for } 0 \le t < 5 \\ 0 & \text{for } t \ge 5. \end{cases}$

36. Solve for the current in the RL circuit of Problem 34 if the initial current is zero and $E(t) = \begin{cases} 0 & \text{for } 0 \le t < 4 \\ Ae^{-t} & \text{for } t \ge 4. \end{cases}$

37. Write the function graphed in Figure 3.26 in terms of the Heaviside function and find its Laplace transform.

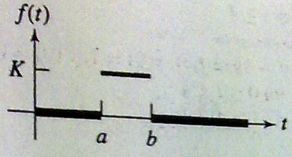

FIGURE 3.26

38. Write the function graphed in Figure 3.27 in terms of the Heaviside function and find its Laplace transform.

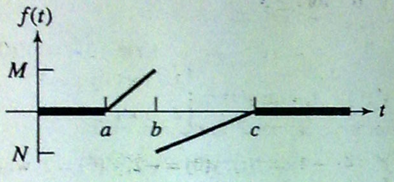

FIGURE 3.27

39. Write the function graphed in Figure 3.28 in terms of the Heaviside function and find its Laplace transform.

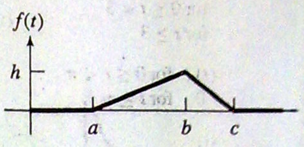

FIGURE 3.28

40. Solve for the current in the RL circuit
if the initial current is zero, $E(t)$ has
$E(t) = \begin{cases} 10 & \text{for } 0 \le t < 2 \\ 0 & \text{for } 2 \le t < 4 \end{cases}$

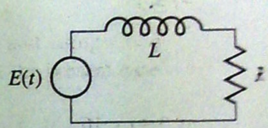

FIGURE 3.29

Hint: See Problem 22 of Section 3.1
transform of a periodic function. You
$I(s) = F(s)/(1+e^{-2s})$ for some $F(s)$.
series to write

$$\frac{1}{1+e^{-2s}} = \sum_{n=0}^{\infty}(-1)^n e^{-2ns}$$

to write $I(s)$ as an infinite series, then
transform term by term by using a shift
Graph the current for $0 \le t < 8$.

3.4 Convolution

In general the Laplace transform of the product of two functions is not the product of their transforms. There is, however, a special kind of product, denoted $f * g$, called the convolution of f with g. Convolution has the feature that the transform of $f * g$ is the product of the transforms of f and g. This fact is called the *convolution theorem*.

DEFINITION 3.6 *Convolution*

If f and g are defined on $[0, \infty)$, then the convolution $f * g$ of f with g is the function defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

for $t \ge 0$.